A BAYESIAN HYPOTHESIS-DECISION PROCEDURE[1]

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1. Introduction

Given the distribution (prior or posterior) of an unknown vector $\boldsymbol{\theta}$ and a positive-definite quadratic loss function l,

$$l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})' L (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) , \tag{1}$$

the optimum estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is, as is well known, $E\boldsymbol{\theta}$. For

$$El(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \operatorname{tr}(\boldsymbol{L}V) + (E\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})' L (E\theta - \hat{\theta}) , \tag{2}$$

where $V = E(\boldsymbol{\theta} - E\boldsymbol{\theta})(\boldsymbol{\theta} - E\boldsymbol{\theta})'$. A decision procedure, based on $E\boldsymbol{\theta}$, is here presented for linear-hypothesis problems in a certain point-estimation context. The method offers the convenience of using moments, which are generally more available than probabilities of events, especially with multidimensional distributions.

2. The decision rule

Suppose a person is contemplating whether to assert that the p-vector $\boldsymbol{\theta}$ lies effectively in a certain r-dimensional linear manifold $S (r < p)$. So to assert is here interpreted as constraining an estimate $\hat{\boldsymbol{\theta}}$ to lie in S. The problem of whether to make the assertion and what estimate to make in either event can be expressed organically as that of minimizing the expectation of a possibly negative loss function of the form,

$$l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) - U_S \cdot S(\hat{\boldsymbol{\theta}}) , \tag{3}$$

where $S(\cdot)$ is the indicator function for the linear manifold S, U_S is the utility of declaring that $\boldsymbol{\theta}$ lies effectively in S; it is the conceptual and practical advantage of the simplified model. Under the Bayes decision rule for the loss function (3), one declares that $\boldsymbol{\theta}$ lies effectively in S if the difference between the minimum, with $\hat{\boldsymbol{\theta}}$ in S, of the ex-

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pectation of $l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ and the unconstrained minimum does not exceed U_S.

We assume that $l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is given by equation (1), and that the coordinates of $\boldsymbol{\theta}$ are so chosen that S contains the origin $\mathbf{0}$. Hence, $\boldsymbol{\theta}$ can be written uniquely as

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1, \tag{4}$$

where $\boldsymbol{\theta}_0$ lies in S and $\boldsymbol{\theta}_1' \boldsymbol{L} \boldsymbol{\zeta} = 0$ for all $\boldsymbol{\zeta}$ in S. (Matrix operators are, of course, available to obtain the projection $\boldsymbol{\theta}_0$ from $\boldsymbol{\theta}$). Similarly, write $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_0 + \hat{\boldsymbol{\theta}}_1$. Then we have

$$l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) = \boldsymbol{L}((\boldsymbol{\theta}_0 - \hat{\boldsymbol{\theta}}_0)) + \boldsymbol{L}((\boldsymbol{\theta}_1 - \hat{\boldsymbol{\theta}}_1)), \tag{5}$$

introducing the notation $\boldsymbol{L}((\boldsymbol{\zeta}))$ for the quadratic form $\boldsymbol{\zeta}' \boldsymbol{L} \boldsymbol{\zeta}$.

The expectation of $l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is minimized when the expectations of both terms on the right-hand side of (5) are minimized. Let $\boldsymbol{\eta}$ be the expectation of $\boldsymbol{\theta}$ and write as in (4), $\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{\eta}_1$. The unconstrained minimum of the expectation of $l(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is attained at $\hat{\boldsymbol{\theta}} = \boldsymbol{\eta}$,

$$El(\boldsymbol{\eta}, \boldsymbol{\theta}) = E\boldsymbol{L}((\boldsymbol{\theta}_0 - \boldsymbol{\eta}_0)) + E\boldsymbol{L}((\boldsymbol{\theta}_1 - \boldsymbol{\eta}_1));$$

and the constrained minimum is attained at $\hat{\boldsymbol{\theta}} = \boldsymbol{\eta}_0$,

$$\begin{aligned} El(\boldsymbol{\eta}_0, \theta) &= E\boldsymbol{L}((\boldsymbol{\theta}_0 - \boldsymbol{\eta}_0)) + E\boldsymbol{L}((\boldsymbol{\theta}_1)) \\ &= El(\boldsymbol{\eta}, \boldsymbol{\theta}) + \boldsymbol{L}((\boldsymbol{\eta}_1)). \end{aligned}$$

Thus one finds oneself comparing $\boldsymbol{L}((\boldsymbol{\eta}_1))$ with U_S. Although the covariance structure of $\boldsymbol{\theta}$ is useful to determine the actual expectation of the loss, the expectation of $\boldsymbol{\theta}$ is the only feature of its distribution formally utilized by the decision rule.

Anscombe [1] has studied many-decision procedures in factor-screening experiments with what in two-decision problems is essentially the loss function, for quadratic l,

$$[l(\hat{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_1) - U_S] \cdot S(\hat{\boldsymbol{\theta}}).$$

The formal decision rule with this loss function is to compare $El(\mathbf{0}, \boldsymbol{\theta}_1)$ with U_S.

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